Rigidity matroids and inductive constructions of graphs and hypergraphs

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Introduction

A d-dimensional *bar-and-joint framework* or *framework* (G,p) is a graph $G=(V,E)$ and a map $p:V\to\mathbb{R}^d$. We say that the framework (G,p) is a *realization* of the graph G in $\mathbb{R}^d$. Two frameworks (G,p) and (G,q) are said to be *equivalent* if $\parallel p(u)-p(v)\parallel=\parallel q(u)-q(v)\parallel$ holds for every edge $uv\in E$. If $\parallel p(u)-p(v)\parallel=\parallel q(u)-q(v)\parallel$ for every pair $u,v\in V$, then (G,p) and (G,q) are *congruent*. We say that a framework (G,p) is *rigid* if there exists an $\varepsilon>0$ such that if (G,q) is equivalent to (G,p) and $\parallel p(v)-q(v)\parallel<\varepsilon$ for all $v\in V$ then (G,q) is congruent to (G,p). Graph G is said to be *(generically) rigid* if it has a rigid generic realization.

An *infinitesimal motion* is an assignment of *infinitesimal velocities* to the vertices of G, $m:V\to\mathbb{R}^d$ satisfying

$$(p(u)-p(v))(m(u)-m(v))=0 \text{ for every pair } u,v \text{ with } uv\in E.$$

An infinitesimal motion is *trivial* if it is an infinitesimal motion of $(K_{|V|},p)$. A framework is *infinitesimally flexible*, if it has a non-trivial infinitesimal motion, otherwise it is *infinitesimally rigid*. For generic frameworks rigidity and infinitesimal rigidity are equivalent [1].

The set of infinitesimal motions of a framework (G,p) is a linear subspace of $\mathbb{R}^{d|V|}$ given by the system of $|E|$ linear equations. If we collect these linear equations into a matrix we get the d-dimensional *rigidity matrix* of the framework. This is the matrix $R_d(G,p)$ of size $|E|\times d|V|$, where, for each edge $uv\in E$, in the row corresponding to uv, the entries in the d columns corresponding to the vertices u and v contain the d coordinates of $(p(u)-p(v))$ and $(p(v)-p(u))$, respectively, and the remaining entries are zeros. $R_d(G,p)$ defines the d-dimensional rigidity matroid of (G,p) on the ground set E by linear independence. We denote this matroid by $\mathcal{R}_d(G)$.

For integers $k\geq 0$ and l, the following definition gives the independent sets of the count matroid $\mathcal{M}_{k,l}(H)$ on the edges of a hypergraph $H=(V,F)$:
A set of hyperedges F is independent if and only if all non-empty subsets F' on vertices V' satisfy $|F'|\leq k|V'|-l$.

Hypergraphs that are independent in $\mathcal{M}_{k,l}(H)$ are called (k,l)*-sparse*. If they also satisfy $|E|=k|V|-l$ then are called (k,l)*-tight* hypergraphs.

The following theorem summarizes the results of Henneberg, Laman [5] and Tay and Whiteley [7].

Theorem 1 *For a graph G, the following are equivalent:*

1. *G is minimally generically rigid in the plane;*
2. *G is $(2,3)$-tight;*
3. *G can be built up from a single edge by a sequence of the following operations:*
   (i) add a new vertex z and connect it to two different existing vertices x and y,

(ii) subdivide an edge uv with a new vertex z and add a new edge between z and an existing vertex w different from u and v.

Operations (i) and (ii) are called the two-dimensional (Henneberg-)0-extension and (Henneberg-)1-extension, respectively.

An *action* of $\mathcal{S}$ on H is a group homomorphism $\rho : \mathcal{S} \to \mathrm{Aut}(H)$. An action ρ is called *free* if $\rho(g)(v) \neq v$ for any $v \in V$ and any non-identity $g \in \mathcal{S}$. We say that a hypergraph H is $(\mathcal{S}, \rho)$*-symmetric* if $\mathcal{S}$ acts on H by ρ. If ρ is clear from the context, we will simply denote $\rho(g)(v)$ by $g \cdot v$ or gv.

Two-dimensional*discrete point groups* (or simply a *point groups*) are classified into two classes, *groups $\mathcal{C}_k$ of k-fold rotations* and *dihedral groups $\mathcal{D}_k$* of order $2k$. For a special case, $\mathcal{D}_1$ consists of a mirror-reflection and the identity.

Suppose that H is $(\mathcal{S}, \rho)$-symmetric for a point group $\mathcal{S}$. A joint-configuration p is said to be $(\mathcal{S}, \rho)$*-symmetric* (or, simply, $\mathcal{S}$-symmetric) if

$$gp(v) = p(gv) \qquad \text{for all } g \in \mathcal{S} \text{ and for all } v \in V(H).$$

Such a pair (H, p) is called an $(\mathcal{S}, \rho)$*-symmetric framework* (or simply an $\mathcal{S}$-symmetric framework or a symmetric framework).

Highly connected rigidity matroids have unique underlying graphs

Let $\mathcal{M}$ be a d-dimensional generic rigidity matroid of some graph G. In Chapter 2 we consider the following problem, posed by Brigitte and Herman Servatius in 2006: is there a (smallest) integer k_d such that the underlying graph G of $\mathcal{M}$ is uniquely determined, provided that $\mathcal{M}$ is k_d-connected? Since the one-dimensional generic rigidity matroid of G is isomorphic to its cycle matroid, a celebrated result of Hassler Whitney implies that $k_1 = 3$. We extend this result by proving that $k_2 \leq 11$. To show this we first prove that if G is highly vertex-connected then it is uniquely determined by its two-dimensional rigidity matroid.

Theorem 2 ([10]) *Let G and H be two graphs and suppose that $\mathcal{R}(G)$ is isomorphic to $\mathcal{R}(H)$. If G is 7-connected then G is isomorphic to H.*

We also show that the bound on the connectivity of G in Theorem 2 cannot be replaced by 5.

Let $\mathcal{M}$ be a matroid on ground set E with rank function r and let k be a positive integer. We say that a partition (X, Y) of E is a *k-separation* if

$$\min\{r(X), r(Y)\} \geq k, \quad \text{and}$$

$$r(X) + r(Y) \leq r(E) + k - 1.$$

The *connectivity* of $\mathcal{M}$, denoted by $\kappa(\mathcal{M})$, is defined to be the smallest integer j for which $\mathcal{M}$ has a j-separation. If $\mathcal{M}$ has no separations at all, we let $\kappa(\mathcal{M}) = r(E)$. We say that $\mathcal{M}$ is *h-connected* if $\kappa(\mathcal{M}) \geq h$ holds.

We also prove that high connectivity of a two-dimensional rigidity matroid implies the high vertex-connectivity of its underlying graph.

Lemma 3 ([10]) *Let $G = (V, E)$ be a graph and suppose that $\mathcal{R}(G)$ is $(2k-3)$-connected for some $k \geq 3$. Then G is k-connected.*

By combining Theorem 2 and Lemma 3 we get the following:

Theorem 4 ([10]) *Let G and H be two graphs and suppose that $\mathcal{R}(G)$ is isomorphic to $\mathcal{R}(H)$. If $\mathcal{R}(G)$ is 11-connected then G is isomorphic to H.*

We also prove the reverse implication: if G is a k-connected graph for some $k \geq 6$ then its two-dimensional rigidity matroid is $(k-2)$-connected. Furthermore, we determine the connectivity of the d-dimensional rigidity matroid of the complete graph K_n, for all pairs of positive integers d, n.

Highly vertex-redundantly rigid graphs

A graph $G = (V, E)$ is called *k-rigid* in $\mathbb{R}^d$ (or *$[k, d]$-rigid* for short) if $|V| \geq k + 1$ and after deleting any set of at most $k - 1$ vertices the resulting graph is rigid in $\mathbb{R}^d$. A k-rigid graph G is called *minimally k-rigid* if the omission of an arbitrary edge results in a graph that is not k-rigid. In Section 3 we give upper and lower bounds for the edge number of minimally highly vertex-redundantly rigid graphs in $\mathbb{R}^d$ for every d.

Theorem 5 ([13]) *Let $G = (V, E)$ be a minimally $[k, d]$-rigid graph. Then*

$$|E| \leq (d + k - 1)|V| - \binom{d+k}{2}.$$

This upper bound is sharp for every $d \geq 2$. The lower bound given in the next theorem is sharp for $[k, d] = [2, d]$ where d is arbitrary, and for $[k, d] = [3, 3]$.

Theorem 6 ([13]) *If a graph $G = (V, E)$ is $[k, d]$-rigid with $|V| \geq d^2 + d + k$ then*

$$|E| \geq d|V| - \binom{d+1}{2} + (k-1)d + \max\left\{0, \left\lceil k - 1 - \frac{d+1}{2} \right\rceil\right\}.$$

The following theorem gives a better lower bound if k is large compared to d.

Theorem 7 ([13]) *Let $k \geq d + 2$ and let $G = (V, E)$ be a $[k, d]$-rigid graph with $|V| \geq d + k$. Then $|E| \geq \left\lceil \frac{d+k-1}{2} |V| \right\rceil$.*

Rigid two-dimensional frameworks with two coincident points

To verify the rigidity of (special families of) generic frameworks it is sometimes useful to consider non-generic realizations of graphs. Motivated by this connection, Jackson and Jordán [4] characterized when a graph has an infinitesimally rigid realization in $\mathbb{R}^2$ in which three given vertices are collinear. An *obstacle* for an ordered triple (x, y, z) of vertices is an ordered triple of tight sets (X, Y, Z) for which $X \cap Y = \{z\}$, $X \cap Z = \{y\}$, and $Y \cap Z = \{x\}$.

Theorem 8 ([4]) *Let $G = (V, E)$ be a minimally rigid graph and let $x, y, z \in V$ be distinct vertices. Then G has an infinitesimally rigid realization (G, p), in which $(p(x), p(y), p(z))$ are collinear if and only if G contains no obstacle for the triple (x, y, z).*

We call a d-dimensional framework (G, p) *U-flat*, for some $U \subseteq V(G)$ with $2 \leq |U| \leq d+1$, if the set $\{p(x) : x \in U\}$ is not affinely independent. Our main result in Chapter 4 is a characterization for the existence of a two-dimensional U-flat realization for a given graph G and $U \subseteq V(G)$ with $|U| = 2$, which completes the solution of the two-dimensional flatness problem.

We need the following definitions. Let $G = (V, E)$ be a graph and let $u, v \in V$ be two distinct vertices of G. A realization (G, p) is called *uv-coincident* if $p(u) = p(v)$ holds. A uv-coincident realization is *uv-generic* if the set of coordinates of the points $\{p(z) : z \in V - v\}$ is algebraically independent over the rationals. Any two uv-coincident uv-generic frameworks (G, p) and (G, p') have the same rigidity matroid. We call this the two-dimensional *uv-rigidity matroid* $\mathcal{R}_{uv}(G) = (E, r_{uv})$ of the graph G. We say that the graph G is *uv-rigid* in $\mathbb{R}^2$ if $r_{uv}(G) = 2|V| - 3$ holds. A set $F \subseteq E$ is said to be *uv-independent* if F is independent in $\mathcal{R}_{uv}(G)$. The graph G is said to be *minimally uv-rigid* if G is uv-rigid and E is uv-independent.

The *0-uv-extension* operation is a 0-extension on a pair a, b with $\{a, b\} \neq \{u, v\}$. The *1-uv-extension* operation is a 1-extension on some edge ab and vertex c for which $\{u, v\}$ is not a subset of $\{a, b, c\}$.

Graph $K_4 - uv$ is defined as follows: it has four vertices, and $u, v \in V(K_4 - uv)$. Every pair of its vertices is connected by an edge except pair (u, v).

Theorem 9 ([9]) *Let $G = (V, E)$ be a graph with $|E| = 2|V| - 3$ and let $u, v \in V$ be distinct vertices. Then G is uv-independent if and only if G can be obtained from $K_4 - uv$ by a sequence of 0-uv-extensions and 1-uv-extensions.*

Let G_{uv} denote the graph obtained from G by contracting the vertex pair u, v into a new vertex z_{uv} (and deleting the resulting loops and parallel copies of edges).Based on the inductive construction we can characterize uv-rigid graphs:

Theorem 10 ([9]) *Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then G is uv-rigid if and only if $G - uv$ and G_{uv} are both rigid.*

Moreover, $r_{uv}(G) = \min\{r(G - uv), r(G_{uv}) + 2\}$ where $r(G)$ denotes the rank function of $\mathcal{R}_2(G)$.

We may also obtain a characterization of minimally uv-rigid graphs which is similar to the obstacle-based characterization for the collinear problem given in Theorem 8.

Theorem 11 ([9]) *Let $G = (V, E)$ be a minimally rigid graph and let $u, v \in V$ be distinct vertices. Suppose that $uv \notin E$. Then the following statements are equivalent:*
(i) G is uv-rigid,
(ii) there is no subgraph $G' = (V', E')$ of G with $\{u, v\} \subseteq V'$ and $|E'| = 2|V'| - (3 + s)$ such that $G' - \{u, v\}$ has at least $s + 2$ components, for $s = 0$ or $s = 1$.

Sparse hypergraphs with applications in combinatorial rigidity

In Chapter 5 our goal is to provide combinatorial tools for attacking problems from rigidity theory in which the underlying combinatorial structure is a hypergraph: projective rigidity, affine rigidity, and scene analysis.

Let $H = (V, E)$ be a $(k+1)$-uniform hypergraph, let j be an integer with $0 \leq j \leq k-1$, and let $v \in V$ be a vertex with $d(v) \geq j$. The *j-extension operation at vertex v* picks j hyperedges $e_1, e_2, ..., e_j$ incident with v, adds a new vertex z to H as well as a new hyperedge e of size $k+1$ incident with both v and z, and replaces e_i by $e_i - v + z$ for all $1 \leq i \leq j$.

2-uniform $(1, 1)$-tight hypergraphs are the trees. The case when $k = 2$ is more complicated, but still not very difficult, so shall focus on 4-uniform $(1, 3)$-tight hypergraphs:

Theorem 12 ([11]) *Let $H = (V, E)$ be a 4-uniform hypergraph. H is $(1, 3)$-tight if and only if it can be obtained from a single hyperedge of size four by a sequence of 0-extensions, 1-extensions, and 2-extensions.*

Projective rigidity on the line

A one-dimensional *projective framework* (H, p) is a pair, where H is a 4-uniform hypergraph and p is a map from $V(H)$ to distinct points of the one-dimensional projective space $\mathbb{P}^1$. A smooth deformation of the framework is called a *flex* if it preserves the cross ratio for each 4-tuple that belongs to the edge set of H and call a framework rigid if it has only trivial flexes (that is, restrictions of a combination of some translation, scaling, and rotation of the space). One may define the infinitesimal rigidity of projective frameworks by considering the rank of the *projective rigidity matrix* $Q(H, p)$ of size $|E| \times |V|$.

A realization (H, p) of a 4-uniform hypergraph $H = (V, E)$ in $\mathbb{P}^1$ is *infinitesimally projectively rigid* if $\operatorname{rank} Q(H, p) = |V| - 3$. We say that $H = (V, E)$ is *generically projectively rigid* in $\mathbb{P}^1$ if there exists an infinitesimally projectively rigid realization of H in $\mathbb{P}^1$. A *minimally generically projectively rigid* hypergraph is a projectively rigid hypergraph with $|E| = |V| - 3$.

George and Ahmed [2] showed that an infinitesimally rigid projective framework is rigid. They also pointed out that a minimally generically projectively rigid hypergraph is $(1,3)$-tight and conjectured that this sparsity condition is also sufficient to guarantee minimal projective rigidity. As an application of our inductive construction (Theorem 12) we proved this conjecture:

Theorem 13 ([11]) *Let $H = (V,E)$ be a 4-uniform hypergraph. Then H is minimally generically projectively rigid in $\mathbb{P}^1$ if and only if H is $(1,3)$-tight.*

Affine rigidity

Gortler et al. [3] introduced the concept of affine rigidity, where affine constraints are imposed on sets of points. A d-dimensional *affine framework* (H,p) is a pair, where H is a hypergraph and p is a map from $V(H)$ to $\mathbb{R}^{\mathrm{d}}$. Roughly speaking, an affine framework (H,p) is *affinely rigid* in $\mathbb{R}^{\mathrm{d}}$ if every other d-dimensional framework (H,q), for which the positions of the vertices in p of each hyperedge $e \in E(H)$ can be mapped to their positions in q by an affine map of $\mathbb{R}^d$, can be obtained by a single affine map of $\mathbb{R}^d$. Gortler et al. [3] define the *strong affinity matrix* of an affine framework (H,p) with $H = (V,E)$, which has size $|E| \times |V|$, and show that the framework is affinely rigid if and only if the rank of this matrix is equal to $|V| - (d+1)$. Thus we may call a hypergraph H *generically affinely rigid* in $\mathbb{R}^{\mathrm{d}}$ if there exists an affinely rigid d-dimensional framework on H. If, in addition, $|E| = |V| - (d+1)$ then H is said to be *minimally generically affinely rigid* in $\mathbb{R}^{\mathrm{d}}$.

There is also a strong connection between affine rigidity and a problem from scene analysis. One can interpret each hyperedge of a planar affine framework as a planar polygon and say that the framework is folded if each vertex can be given a third coordinate, such that, in the resulting three dimensional drawing, each polygon remains planar, and the faces do not all lie in a single plane. Whiteley [8] showed that a framework has a non-trivial lifting if and only if it is not affinely rigid or equivalently $(1, 3+1)$-tight.

Thus an immediate corollary of Theorem 12 is an inductive construction of the 4-uniform minimally generically affinely rigid hypergraphs in the plane.

Theorem 14 ([11]) *Let $H = (V,E)$ be a 4-uniform hypergraph. Then H is minimally generically affinely rigid in $\mathbb{R}^2$ if and only if it can be obtained from a single hyperedge of size four by a sequence of 0-extensions, 1-extensions, and 2-extensions.*

Gain-sparsity and Symmetry-forced Rigidity in the Plane

Chapter 6 deals with finite bar-and-joint frameworks with point group symmetry in the *symmetry-forced* setting and extends Laman's classical theorem as well as its matroidal background and algorithmic implications, to planar frameworks with rotational or dihedral symmetry, assuming that the joint positions are as generic as possible subject to the symmetry conditions. In our symmetry-forced setting, a framework is said to be *symmetry-forced flexible* if it has a non-trivial *symmetric* infinitesimal motion.

Let $G = (V, E)$ be a directed multigraph, and let $\mathcal{S}$ be a group. An $\mathcal{S}$-*gain graph* (G, ϕ) is a pair, in which each edge is associated with an element of $\mathcal{S}$ by a *gain function* $\phi : E \to \mathcal{S}$. For $X \subseteq E(G)$ and $v \in V(G)$, $\pi_1(X, v)$ denotes the set of closed walks starting at v and using only edges of X. The *gain* of W is defined as $\phi(W) = \phi(e_1) \cdot \phi(e_2) \cdots \phi(e_k)$ if each edge is oriented in the forward direction through W, and for a backward edge e_i we replace $\phi(e_i)$ with $\phi(e_i)^{-1}$ in the product. The subgroup induced by X relative to v is defined as $\langle X \rangle_{\phi,v} = \{\phi(W) \mid W \in \pi_1(X, v)\}$.

An edge subset $F \subseteq E(G)$ is called *balanced* if $\langle F \rangle_{\psi,v}$ is trivial for every $v \in V(F)$. In the same way, an edge subset $F \subseteq E(G)$ is called *cyclic* if $\langle F \rangle_{\psi,v}$ is cyclic for every $v \in V(F)$.

Let H be an $\mathcal{S}$-symmetric graph. The *quotient graph* $H/\mathcal{S}$ of H is a gain graph on the set $V(H)/\mathcal{S}$ of vertex orbits, together with the set $E(H)/\mathcal{S}$ of edge orbits as the edge set. To construct $H/\mathcal{S}$, we arbitrarily choose a vertex v as a representative vertex from each vertex orbit. Then, each orbit is written by $\mathcal{S}v = \{gv \colon g \in \mathcal{S}\}$. An edge orbit connecting $\mathcal{S}u$ and $\mathcal{S}v$ in $H/\mathcal{S}$ can be written by $\{\{gu, ghv\} \colon g \in \mathcal{S}\}$ for a unique $h \in \mathcal{S}$. We then orient the edge orbit from $\mathcal{S}u$ to $\mathcal{S}v$ in $H/\mathcal{S}$ and assign to it the gain h.

Cyclic point groups

Definition 15 *Let (G, ϕ) be an $\mathcal{S}$-gain graph with a graph $G = (V, E)$ and a group $\mathcal{S}$. An edge set $X \subseteq E$ is called* (2, 3)-gain-sparse *(or* (2, 3)-g-sparse *for short) if*

- $|F| \leq 2|V(F)| - 3$ *for every nonempty balanced $F \subseteq X$;*
- $|F| \leq 2|V(F)| - 1$ *for every nonempty unbalanced $F \subseteq X$.*

(G, ϕ) is called (2, 3)-g-sparse *if so is E, and it is called* maximum (2, 3)-g-tight *if it is (2, 3)-g-sparse with $|E| = 2|V| - 3$.*

(2, 3)-gain-sparse edge sets of (G, ϕ) are the independent sets of a matroid.

We prove an constructive characterization for maximum (2,3)-g-tight graphs. First we define three operations, called *extensions*, that preserve (2, 3)-g-sparsity.

The *0-extension* adds a new vertex v and two new non-loop edges e_1 and e_2 to G such that the new edges are incident to v and the other endvertices are two not necessarily distinct vertices of $V(G)$. If e_1 and e_2 are not parallel then their labels can be arbitrary. Otherwise the labels are assigned such that $\phi(e_1) \neq \phi(e_2)$, assuming that e_1 and e_2 are directed to v.

The *1-extension* first chooses an edge e and a vertex z, where e may be a loop and z may be an endvertex of e. It subdivides e, with a new vertex v and new edges e_1, e_2 such that the tail of e_1 is the tail of e and the tail of e_2 is the head of e. The labels of the new edges are assigned such that $\phi(e_1) \cdot \phi(e_2)^{-1} = \phi(e)$. The 1-extension also adds a third edge $zv = e_3$. The label of e_3 is assigned so that it is *locally unbalanced*, i.e., every two-cycle $e_i e_j$, if exists, is unbalanced.

The *loop 1-extension* adds a new vertex v to G and connects it to a vertex $z \in V(G)$ by a new edge with any label. It also adds a new loop l incident to v with $\phi(l) \neq \mathrm{id}$.

Theorem 16 ([12]) *An $\mathcal{S}$-gain graph (G, ϕ) is maximum (2,3)-g-tight if and only if it can be built up from an $\mathcal{S}$-gain graph with one vertex and an unbalanced loop incident to it with a sequence of 0-extensions, 1-extensions, and loop-1-extensions.*

Then using this construction we can get the following characterization:

Theorem 17 ([12]) *Let $\mathcal{C} \subset \mathcal{O}(\mathbb{R}^2)$ be a cyclic point group, that is, either a group of k-fold rotations or a group of a reflection, and let (H, p) be a generic $(\mathcal{C}, \rho)$-symmetric framework in the plane with a free action ρ. Then (H, p) is symmetry-forced infinitesimally rigid if and only if the quotient $\mathcal{C}$-gain graph contains a spanning maximum $(2, 3)$-g-tight subgraph.*

Dihedral point groups $\mathcal{D}_k$ with odd k

We next move to non-cyclic point groups, that is, dihedral groups of order $2k$ that we denote by $\mathcal{D}_k$ (or simply by $\mathcal{D}$).

Definition 18 *Let (G, ϕ) be a $\mathcal{D}$-gain graph. An edge set $X \subseteq E(G)$ is called* $\mathcal{D}$-sparse *if*

- $|F| \leq 2|V(F)| - 3$ *for every nonempty balanced $F \subseteq X$;*
- $|F| \leq 2|V(F)| - 1$ *for every nonempty cyclic $F \subseteq X$;*
- $|F| \leq 2|V(F)|$ *for every $F \subseteq X$.*

(G, ϕ) is said to be $\mathcal{D}$-sparse *if so is $E(G)$, and it is called* maximum $\mathcal{D}$-tight *if it is $\mathcal{D}$-sparse with $|E(G)| = 2|V(G)|$.*

$\mathcal{D}$-sparse edge sets of (G, ϕ) are the independent sets of a matroid.

To obtain an constructive characterization for maximum $\mathcal{D}$-tight graphs we need some operations that add degree four vertices.

By developing a constructive characterization for maximum $\mathcal{D}$-tight gain graphs we can prove the following:

Theorem 19 ([12]) *Let $\mathcal{D}_k$ be a dihedral group with odd $k \geq 3$, and (H, p) be a generic $(\mathcal{D}_k, \rho)$-symmetric framework with a free action ρ. Then (H, p) is symmetry-forced infinitesimally rigid if and only if the quotient gain graph contains a spanning maximum $\mathcal{D}_k$-tight subgraph.*

Lifting symmetric pictures to polyhedral scenes

In the final chapter we develop the $\mathcal{C}_3$-symmetric version of the hypergraph construction given in Chapter 5 and then use it to characterize $\mathcal{C}_3$-symmetric 2-scenes. We first give the formal definitions of scene analysis.

A *(polyhedral) incidence structure* S is an abstract set of vertices V, an abstract set of faces F, and a set of incidences $I \subseteq V \times F$. (We can think of an incidence structure as a hypergraph.)

A 2-*picture* is an incidence structure S together with a corresponding map $r : V \to \mathbb{R}^2$, $r_i = (x_i, y_i)$, and is denoted by $S(r)$.

A 3-*scene* $S(p, P)$ is an incidence structure $S = (V, F; I)$ together with a pair of maps, $p : V \to \mathbb{R}^3$, $p_i = (x_i, y_i, z_i)$, and $P : F \to \mathbb{R}^3$, $P^j = (A^j, B^j, C^j)$, such that for each $(i, j) \in I$ we have $A^j x_i + B^j y_i + z_i + C^j = 0$. (We assume that no hyperplane is vertical, i.e., is parallel to the vector $(0, \dots, 0, 1)^T$.)

A *lifting* of a (2)-picture $S(r)$ is a 3-scene $S(p, P)$, with the vertical projection $\Pi(p) = r$. That is, if $p_i = (x_i, y_i, z_i)$, then $r_i = (x_i, y_i) = \Pi(p_i)$.

A lifting $S(p, P)$ is *trivial* if all the faces lie in the same plane. Further, $S(p, P)$ is *folded* (or *non-trivial*) if some pair of faces have different planes, and is *sharp* if each pair of faces sharing a vertex have distinct planes. A picture which has no non-trivial lifting is called *flat* (or *trivial*). A picture with a non-trivial lifting is called *foldable*.

The *lifting matrix* for a picture $S(r)$ is the $|I| \times (|V| + 3|F|)$ coefficient matrix $M(S, r)$ of the system of equations for liftings of a picture $S(r)$.

As it can be seen easily that it suffices to consider 4-uniform structures only, we will focus only on the 4-uniform case.

Theorem 20 ([8], special case) *A generic 2-picture of a 4-uniform S has independent rows in the lifting matrix if and only if S is (1,3)-sparse.*

Our goal is to prove the $\mathcal{C}_3$-symmetric version of Theorem 20.

A vertex v of S is said to be *fixed* by C_3 if $C_3 v = v$. Similarly, a face $f = \{v_1, \dots, v_m\}$ of S is said to be *fixed* by C_3 if $C_3 f = f$, i.e., if $C_3(\{v_1, \dots, v_m\}) = \{v_1, \dots, v_m\}$. Finally, an incidence (i, j) of S is said to be *fixed* by C_3 if $C_3((i, j)) = (i, j)$. V_3, F_3, I_3 denotes the set of fixed vertices, faces and incidences, respectively. In [6] Schulze showed that if a $\mathcal{C}_3$-symmetric symmetry-generic picture of a 4-uniform incidence structure with $|F| = |V| - 3$ is minimally flat then S is $(1, 3)$-tight and $|I_3| = |V_3|$ holds. The main result of Chapter 7 is that these two necessary conditions are sufficient for the existence of a minimally $\mathcal{C}_3$-symmetric flat symmetry-generic picture:

Theorem 21 ([14]) *A $\mathcal{C}_3$-symmetric and symmetry-generic picture of a 4-uniform incidence structure S with $|F| = |V| - 3$ has independent rows in the lifting matrix if and only if S is (1,3)-tight and $|V_3| = |I_3|$.*

References

[1] L. Asimow and B. Roth, The rigidity of graphs. *Trans. Amer. Math. Soc.*, 245:279-289, 1978.

[2] S. George and F. Ahmed, Rigidity in the one-dimensional projective space, manuscript. See `http://www.math.columbia.edu/ dpt/RigidityREU/AG11Rigidity1DProj.pdf`

[3] S.J. Gortler, C. Gotsman, L. Liu, and D.P. Thurston, On affine rigidity, arXiv:1011.5553v3, August 2013.

[4] B. Jackson and T. Jordán, Rigid two-dimensional frameworks with three collinear points, *Graphs and Combinatorics* (2005) 21:427-444.

[5] G. Laman, On graphs and rigidity of plane skeletal structures, *J. Engineering Math.* 4 (1970), 331-340.

[6] B. Schulze, Lifting symmetric pictures, manuscript, 2013.

[7] T.-S. Tay and W. Whiteley, Generating isostatic frameworks, *Structural Topology* 11, 1985, pp. 21-69.

[8] W. Whiteley, A matroid on hypergraphs with appliations in scene analysis and geometry, *Discrete and Computational Geometry* 4, 1 (1989) 75-95.

The thesis is based on the following publications

[9] Zs. Fekete, T. Jordán, V. E. Kaszanitzky, Rigid two-dimensional frameworks with two coincident points, *Graphs and Combinatorics*, May 2015, Volume 31, Issue 3, pp 585-599

[10] T. Jordán, V. E. Kaszanitzky, Highly connected rigidity matroids have unique underlying graphs, *European Journal of Combinatorics*, Volume 34, Issue 2, February 2013, Pages 240-247.

[11] T. Jordán, V. E. Kaszanitzky, Sparse hypergraphs with applications in combinatorial rigidity, *Discrete Applied Mathematics*, Volume 185, 20 April 2015, Pages 93-101

[12] T. Jordán, V.E. Kaszanitzky, and S. Tanigawa, Gain-sparsity and symmetry-forced rigidity in the plane. EGRES technical report, TR-2012-17, 2012.

[13] V.E. Kaszanitzky, Cs. Király, On Minimally Highly Vertex-Redundantly Rigid Graphs *Graphs and Combinatorics*, DOI 10.1007/s00373-015-1560-3

[14] V.E. Kaszanitzky, B. Schulze, Lifting symmetric pictures to polyhedral scenes, manuscript, 2013.